ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Sound and Vibration 265 (2003) 765–782

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

A new analytical technique for vibration analysis of non-proportionally damped beams

S. Sorrentino, S. Marchesiello*, B.A.D. Piombo

Dipartimento di Meccanica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

Received 23 May 2002; accepted 7 August 2002

Abstract

Vibrating linear mechanical systems, in particular continuous systems, are often modelled considering proportional damping distributions only, although in many real situations this simplified approach does not describe the dynamics of the system with sufficient accuracy. In this paper an analytical method is given to take into account the effects of a more general viscous damping model, referred to as non-proportional damping, on a class of vibrating continuous systems. A state-form expansion applied in conjunction with a transfer matrix technique is adopted to extract the eigenvalues and to express the eigenfunctions in analytical form, i.e., complex modes corresponding to non-synchronous motions. Numerical examples are included in order to show the efficiency of the proposed method; non-proportional damping distributions of different type, such as internal and external lumped or distributed viscous damping, are tested on non-homogeneous Euler–Bernoulli beams in bending vibration with different boundary conditions. Finally, a discussion on root locus diagrams behaviour and on modal damping ratio significance for non-proportionally damped systems is presented.

1. Introduction

The effects of non-proportional damping distributions on vibrating linear mechanical systems have been not exhaustively studied in terms of modal analysis, especially with regard to continuous systems. In fact, continuous systems are often modelled considering proportional damping distributions only, which carry little further analytical and computational effort in addition to the undamped case analysis, since the mode shape remain unchanged. However in many real situations, the proportional damping assumption is not valid and this simplified

*Corresponding author. Tel.: +39-11-564-6950; fax: +39-11-564-6999.
E-mail address: marchesiello@polito.it (S. Marchesiello).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(02)01560-2

approach does not describe the dynamics of the system with sufficient accuracy. In particular, the efforts of many authors were devoted to quantify the non-proportionality, giving a measure of the difference between proportional and non-proportional viscous damping models for a multi-degree-of-freedom systems [1], but such indexes, unfortunately, give no information in order to predict the effects of a known damping distribution on the dynamic behaviour.

The problem of vibrating, non-homogeneous, continuous systems, especially Euler–Bernoulli beams, seems to be of some interest in engineering, since a large number of papers on this subject have been published in recent years. For example, the following particular cases have been considered: transverse vibrations of an Euler–Bernoulli beam carrying two particles in-span [2], free vibrations of stepped beams with intermediate elastic supports [3], free vibrations of axially loaded beams with intermediate elastic supports [4], continuous structures with lumped spring and masses [5], and straight beams with different boundary conditions [6].

Conversely, only few authors have included non-proportional damping in their analysis of continuous vibrating systems. These include a method to compute the eigenvalues of a homogeneous beam with lumped mass, stiffness and damping elements is presented in Ref. [7], applying the Laplace transform technique; complex modes of non-homogeneous vibrating beams with lumped or piecewise constant distributed viscous damping are computed in Ref. [8], by dividing the beam into homogeneous segments and then setting the boundary conditions in a matrix, whose dimensions proportionally increase with the number of segments; a similar approach, with a more general substructure coupling procedure, is presented in Refs. [9,10].

The method proposed in this paper, starting from a partition of the continuous system in homogeneous substructures or segments, presents a different approach based on the reduction of the differential equations order in conjunction with a transfer matrix technique. Consequently, it can be easily applied to a large number of continuous vibrating systems with non-proportional damping, provided that closed-form solutions of the undamped case for each homogeneous segment are known. Moreover, the proposed approach leads to an easy computer implementation and presents a high computational efficiency, due to the invariance of the matrix dimensions with respect to the number of segments considered. The particular case of flexural vibrations of Euler–Bernoulli non-homogeneous beams with different non-proportional internal and external damping distributions and various lumped constraints is considered here. Actually, only few coefficient adjustments would be required to analyze the problem of strings, rods, shafts or Timoshenko beams with viscous or more complicated damping models. Some numerical examples are included, showing the effects of particular cases of internal and external non-proportional damping on eigenvalues, eigenfunctions and root locus diagrams.

2. Method of analysis

In this section an analytical technique is proposed to solve the free vibration problem for non-proportionally damped Euler–Bernoulli non-homogeneous beams. In particular, two different damping models are considered, that is, internal viscous damping and external lumped or distributed viscous damping. By separating spatial and time variables, the classical boundary value problem reduces to a differential eigenvalue problem. Assuming the beam to be a sequence of homogeneous segments leads to a set of ordinary differential equations, one for each segment, which can be solved applying appropriate boundary conditions. Finally, a state-form expansion applied in conjunction with a transfer matrix technique (which is a modification of the classical transfer matrix technique [11]) make it possible to extract the eigenvalues and to express the eigenfunctions in analytical form.

2.1. The boundary value problem

The partial differential equation of motion for a distributed viscously damped system can be expressed in the operator form

$$M\left[\frac{\partial^2}{\partial t^2}w(\mathbf{x},\, t)\right] + C\left[\frac{\partial}{\partial t}w(\mathbf{x},\, t)\right] + K[w(\mathbf{x},\, t)] = f(\mathbf{x},\, t), \quad \mathbf{x}\in D, \tag{1}$$

where M, C, K are linear homogeneous differential operators [12], and are referred to as mass operator, damping operator and stiffness operator, respectively, f is the external force density (also including all non-conservative forces other than viscous damping ones), w and $\mathbf{x}$ are the displacement and the spatial variable in the domain of extension D, respectively. Associated with the differential equation (1), appropriate boundary conditions must be satisfied by the solution w at every point of the boundary of the domain D.

In the special case of an Euler–Bernoulli beam in bending vibration, the mass operator and the stiffness operator consist of

$$M = m(x), \tag{2}$$

$$K = \frac{\partial^2}{\partial x^2}\left[k(x)\frac{\partial^2}{\partial x^2}\right], \tag{3}$$

where $m(x)$ is the mass per unit length of beam and $k(x) = EI(x)$ is the bending stiffness, or flexural rigidity, in which E is Young's modulus of elasticity and I is the area moment of inertia. The damping operator can be expressed as

$$C = \frac{\partial^2}{\partial x^2}\left[c_{in}(x)\frac{\partial^2}{\partial x^2}\right], \tag{4}$$

which is the case of internal damping (according with the Kelvin–Voigt model, used in conjunction with the assumption that cross-sectional areas remain planar during deformation [12]) or, simply, as

$$\mathrm{C} = c(x), \tag{5}$$

which is the case of external distributed viscous damping (note that the two distributions $c_{in}(x)$ and $c(x)$ are dimensionally different).

If the damping operator can be expressed as a linear combination of the mass operator and stiffness operator, damping is said to be proportional. In this paper the more general case in which damping results to be non-proportional is considered.

The partial differential equation of motion (1) for an Euler–Bernoulli beam in bending vibration with external distributed viscous damping under the distributed transverse force

$f(x, t)$ is

$$m(x)\frac{\partial^2}{\partial t^2}w(x, t) + c(x)\frac{\partial}{\partial t}w(x, t) + \frac{\partial^2}{\partial x^2}\left[k(x)\frac{\partial^2}{\partial x^2}w(x, t)\right] = f(x, t). \tag{6}$$

Two boundary conditions must be satisfied at $x = 0$ and l (length of the beam). In the simple cases of clamped end, pinned end and free end, they are

$$\text{clamped end} \quad \begin{cases} w(x, t) = 0, \\ \dfrac{\partial}{\partial x}w(x, t) = 0, \end{cases} \tag{7}$$

$$\text{pinned end} \quad \begin{cases} w(x, t) = 0, \\ k(x)\dfrac{\partial^2}{\partial x^2}w(x, t) = 0, \end{cases} \tag{8}$$

$$\text{free end} \quad \begin{cases} k(x)\dfrac{\partial^2}{\partial x^2}w(x, t) = 0, \\ \dfrac{\partial}{\partial x}\left[k(x)\dfrac{\partial^2}{\partial x^2}w(x, t)\right] = 0. \end{cases} \tag{9}$$

2.2. The differential eigenvalue problem

In the absence of external forces, $f(x, t) = 0$, the partial differential equation of motion (6) reduces to

$$m(x)\frac{\partial^2}{\partial t^2}w(x, t) + c(x)\frac{\partial}{\partial t}w(x, t) + \frac{\partial^2}{\partial x^2}\left[k(x)\frac{\partial^2}{\partial x^2}w(x, t)\right] = 0, \tag{10}$$

which, in conjunction with appropriate boundary conditions, describes the free bending vibration of the beam. In the following, the existence of solutions of Eq. (10) will be explored by separating the variables in the form

$$w(x, t) = \phi(x)q(t), \tag{11}$$

where ϕ only depends on the spatial position and q only depends on time.

The above separation of variables leads to the reduction of the boundary value problem into a differential eigenvalue problem [12]. Introducing Eq. (11) in Eq. (1), defining the state vector

$$\mathbf{w}(x, t) = \begin{Bmatrix} w(x, t) \\ \dot{w}(x, t) \end{Bmatrix}, \tag{12}$$

(the dot denoting derivative with respect to time) and the linear homogeneous differential operators

$$L_1 = \begin{bmatrix} 0 & M \\ -K & -C \end{bmatrix}, \quad L_2 = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, \tag{13}$$

it is possible to rewrite the differential equation of motion (1) in the standard state-form, consisting of two first order ordinary differential equations. As is known, the solution of the homogeneous system follows from the associated eigenvalue problem, which can be stated in the canonical form

$$L_1[\mathbf{w}] = sL_2[\mathbf{w}], \tag{14}$$

where the generally complex constant s has to be determined so that Eq. (14) admits non-trivial solutions satisfying the boundary conditions [12].

In order to acknowledge the effects of non-proportional viscous damping, the differential eigenvalue problem (14) will be solved for the special case (of some interest in engineering) in which $m(x)$, $c(x)$ and $k(x)$ can be considered piecewise constant on D (see Fig. 1).

Dividing the beam into N segments of length $\Delta x_p = x_p—x_{p-1}$ (where $x_0 = 0$, $x_N = l$), and assuming $m(x)$, $c(x)$ and $k(x)$ constant on each segment, Eq. (14) reduces to a set of N fourth order ordinary differential equations with constant coefficients of the type

$$\phi_p^{\mathrm{IV}}(x) = -\sigma_p \phi_p(x), \tag{15}$$

(the order of derivative with respect to the spatial variable is denoted by a roman number) with appropriate boundary conditions, where

$$\sigma_p = (m_p s^2 + c_p s)k_p^{-1}, \tag{16}$$

which holds for external distributed damping, or

$$\sigma_p = m_p s^2 (c_{in,p} s + k_p)^{-1}, \tag{17}$$

which holds for internal damping. Note that more complicated damping laws, even involving fractional derivatives, could be easily taken into account simply by modifying the definition of σ_p in function of s. In any case, s is obviously the same for every segment.

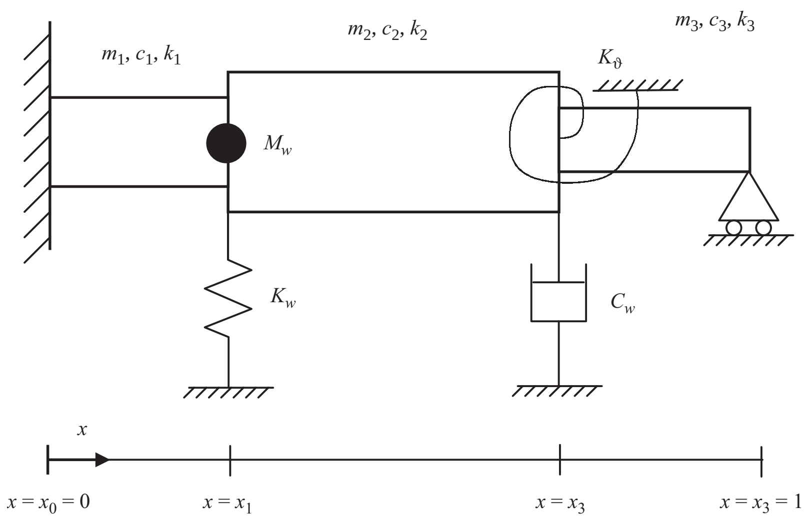

Fig. 1. Non-homogeneous Euler–Bernoulli beam with non-proportional damping and different constraints.

Once again it is convenient to put the problem in state-form, thus Eq. (15) becomes

$$\mathbf{y}_p^{\mathrm{I}}(x) = \mathbf{S}_p \mathbf{y}_p(x), \tag{18}$$

where $\mathbf{y}$ and $\mathbf{S}_p$ correspond to a state vector and to the transpose of a companion matrix, respectively, i.e.,

$$\mathbf{y}(x) = \begin{Bmatrix} \phi^{\mathrm{III}}(x) \\ \phi^{\mathrm{II}}(x) \\ \phi^{\mathrm{I}}(x) \\ \phi(x) \end{Bmatrix}, \quad \mathbf{S}_p = \begin{bmatrix} 0 & 0 & 0 & -\sigma_p \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{19}$$

The four eigenvalues (functions of s) of the companion matrix are

$$\begin{aligned} \lambda_{p1} &= \sqrt[4]{-\sigma_p} = a_p, \\ \lambda_{p2} &= -\sqrt[4]{-\sigma_p} = -a_p, \\ \lambda_{p3} &= \mathrm{i}\sqrt[4]{-\sigma_p} = \mathrm{i}a_p, \\ \lambda_{p4} &= -\mathrm{i}\sqrt[4]{-\sigma_p} = -\mathrm{i}a_p, \end{aligned} \tag{20}$$

where a_p corresponds to a transmission factor; so the solution of Eq. (18) is

$$\mathbf{y}_p(x) = \boldsymbol{\Phi}_p \mathrm{e}^{\boldsymbol{\Lambda}_p x} \mathbf{c}_p, \tag{21}$$

where $\boldsymbol{\Phi}_p$ is the pth segment eigenvectors matrix, $\boldsymbol{\Lambda}_p$ is the pth segment eigenvalues matrix and $\mathbf{c}_p$ is the pth segment constants vector.

2.3. The boundary conditions

The boundary conditions at the ends of the beam can be written in the form

$$\begin{aligned} \mathbf{B}_{e0} \mathbf{y}_1(0) &= \mathbf{0}, \\ \mathbf{B}_{el} \mathbf{y}_N(l) &= \mathbf{0}, \end{aligned} \tag{22}$$

where $\mathbf{B}_e$ are 2×4 matrices depending on the kind of constraints. For a clamped end, a pinned end or a free end, matrices $\mathbf{B}_e$ are simply

$$\underset{\text{clamped}}{\mathbf{B}_e = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}, \quad \underset{\text{pinned}}{\mathbf{B}_e = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}, \quad \underset{\text{free}}{\mathbf{B}_e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}, \tag{23}$$

which become a little more complicated if at the ends of the beam there are other external constraints, such as lumped inertia, damping or stiffness elements.

Moreover, it is necessary to introduce appropriate constraint conditions at the ends of each segment. The $N-1$ boundary conditions between the N beam segments can be written as

$$\mathbf{y}_p(x_{p-1}) = \mathbf{B}_{p-1} \mathbf{y}_{p-1}(x_{p-1}), \quad p = 2, \ldots, N, \tag{24}$$

where $\mathbf{B}_{p-1}$ are 4×4 matrices obtained by imposing the continuity of displacement, rotation, moment and shear in $x = x_{p-1}$. $\mathbf{B}_{p-1}$ can be written in the following form:

$$\mathbf{B}_{p-1} = \begin{bmatrix} b_p^{-1} b_{p-1} & 0 & 0 & -b_p^{-1} r_{p-1}^{(w)} \\ 0 & b_p^{-1} b_{p-1} & b_p^{-1} r_{p-1}^{(\vartheta)} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{25}$$

where b_p and b_{p-1} depend on internal stiffness and damping, i.e.,

$$\begin{aligned} b_p &= k_p && \text{(segment } p \text{ undamped or with external } \textit{distributed} \text{ damping)}, \\ b_p &= c_{in,p} s + k_p && \text{(segment } p \text{ with internal damping)}, \end{aligned} \tag{26}$$

while $r^{(w)}$ and $r^{(\vartheta)}$ depend on external constraints.

Note that $\mathbf{B}_{p-1} = \mathbf{I}$ if the following conditions are all satisfied:

- no external constraints at x_{p-1};
- segments p and p—1 without *internal* damping;
- $k_p = k_{p-1}$.

If at x_{p-1} there are external constraints, such as lumped inertia, damping or stiffness elements, then, in accordance with Fig. 1,

$$\begin{aligned} r^{(w)} &= M_w s^2 + C_w s + K_w, \\ r^{(\vartheta)} &= K_\vartheta, \end{aligned} \tag{27}$$

where M_w is a lumped mass, C_w is a lumped damping, K_w is a lumped linear stiffness, and K_ϑ is a lumped rotational stiffness.

Taking into account Eqs. (21) and (24), it is not difficult to derive the relation

$$\mathbf{y}_p(x) = \Phi_p e^{\Lambda_p(x - x_{p-1})} \Phi_p^{-1} \mathbf{B}_{p-1} \mathbf{y}_{p-1}(x_{p-1}), \tag{28}$$

for $x_{p-1} < x \leqslant x_p$, where $p = 1, \ldots N$, $\mathbf{B}_0 = \mathbf{I}$ and the pth segment eigenvectors matrix and its inverse, written as functions of a_p, have the form

$$\Phi_p = \begin{bmatrix} a_p^3 & -a_p^3 & -\mathrm{i}a_p^3 & \mathrm{i}a_p^3 \\ a_p^2 & a_p^2 & -a_p^2 & -a_p^2 \\ a_p & -a_p & \mathrm{i}a_p & -\mathrm{i}a_p \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \Phi_p^{-1} = \frac{1}{4} \begin{bmatrix} a_p^{-3} & a_p^{-2} & a_p^{-1} & 1 \\ -a_p^{-3} & a_p^{-2} & -a_p^{-1} & 1 \\ \mathrm{i}a_p^{-3} & -a_p^{-2} & -\mathrm{i}a_p^{-1} & 1 \\ -\mathrm{i}a_p^{-3} & -a_p^{-2} & \mathrm{i}a_p^{-1} & 1 \end{bmatrix}. \tag{29}$$

Moreover, Eq. (28) yields

$$\mathbf{y}_p(x_p) = \underbrace{\prod_{i=p}^{1} [\Phi_i e^{\Lambda_i(x_i - x_{i-1})} \Phi_i^{-1} \mathbf{B}_{i-1}]}_{\Pi_p(x_p)} \mathbf{y}_1(0), \tag{30}$$

so it is possible to express $\mathbf{y}(l)$ as a function of $\mathbf{y}(0)$, i.e.,

$$\mathbf{y}_N(l) = \Pi_{\mathrm{N}}(l) \mathbf{y}_1(0), \tag{31}$$

and the boundary conditions (22) become

$$\begin{cases} \mathbf{B}_{e0}\mathbf{y}_1(0) = \mathbf{0} \\ \mathbf{B}_{el}\Pi_N(l)\mathbf{y}_1(0) = \mathbf{0} \end{cases} \quad \text{or, equivalently} \quad \begin{cases} \mathbf{B}_{e0}\Phi_1\mathbf{c}_1 = \mathbf{0}, \\ \mathbf{B}_{el}\Pi_N(l)\Phi_1\mathbf{c}_1 = \mathbf{0}. \end{cases} \tag{32}$$

Eq. (32) forms a set of four linear homogeneous algebraic equations in four unknowns c_{1i} and can be rewritten in the form

$$\Theta\mathbf{c}_1 = \mathbf{0}. \tag{33}$$

Remembering that the elements of the coefficient matrix $\mathbf{\Theta}$ depend on the unknown eigenvalue s, and that Eq. (33) possesses a non-trivial solution if and only if the determinant of the coefficient matrix is zero, then the solutions of

$$\det[\Theta] = 0, \tag{34}$$

are the solutions of the differential eigenvalue problem (14). In case of underdamped continuous systems, they form a infinite set of complex conjugate pairs s_n, s_n^* of discrete values. Each pair characterizes a mode and is related with a pair of eigenfunctions, which result step-defined from Eq. (28), except for a global complex constant.

The solution $w_n(x, t)$ related to the nth mode (in the following referred to as modal displacement) must be real and it can be expressed in the form

$$w_n(x, t) = \gamma_n\phi_n(x)e^{s_n t} + \gamma_n^*\phi_n^*(x)e^{s_n^* t} = 2\mathrm{Re}[\gamma_n\phi_n(x)e^{s_n t}], \tag{35}$$

or, equivalently,

$$w_n(x, t) = 2e^{-\Gamma_n t}\{\mathrm{Re}\,[\gamma_n\phi_n(x)]\cos\Omega_n t - \mathrm{Im}\,[\gamma_n\phi_n(x)]\sin\Omega_n t\}, \tag{36}$$

where γ_n is an indeterminate scaling factor, generally complex, which depends on the initial conditions,

$\Omega_n = |\mathrm{Im}[s_n]|$ is the modal frequency of damped free vibration,
$\Gamma_n = -\mathrm{Re}\,[s_n] = -\mathrm{Re}[s_n^*]$ is the modal damping factor.

It is very important to note that, in general, the well-known relations $\Gamma_n = \zeta_n\omega_n$ and $\Omega_n = \omega_n\sqrt{1-\zeta_n^2}$, where ω_n is the modal natural frequency and ζ_n is the modal damping ratio, are valid only if damping is proportional. Moreover, if damping is proportional, $\arg[\phi(x)]$ is constant, whilst in general for non-proportional damping $\arg[\phi(x)]$ varies with respect to the spatial variable. As a consequence, from Eq. (36) it is clear that if damping is non-proportional, then the system does not execute synchronous motions, which are characterized by the fact that the ratio of the displacements corresponding to different points is constant with respect to time [12].

It should finally be noted that mathematically Eq. (36) provides a solution in the classical sense (i.e., four time continuously differentiable in D) everywhere, except in a finite subset of D (i.e., $x = x_p$, with $p = 1, \ldots, N—1$); here is weak as a consequence of the discontinuities introduced in functions $m(x)$, $c(x)$ and $k(x)$, which have been assumed to be piecewise constant on D.

3. Numerical examples

Three numerical examples are presented as applications of the proposed method, in order both to validate it by comparisons to other less general methods and to show its reliability in problems involving non-proportional damping. Moreover, it should be noted that the proposed method has also been successfully applied by the authors to solve a number of particular cases, proposed in several papers concerning non-homogeneous continuous systems [2–4,7].

In all numerical cases studied, the proposed method has performed well in terms of computational time, due to the reduced dimensions of the matrices involved in the numerical procedure. In fact, (4×4) matrices are always needed for the method to be applied to continuous beams of any complexity. Most of the computational effort is only devoted to a zero finding routine in the complex domain, involved in Eq. (34). This problem has been solved by means of the secant method [13] applied to a real function of complex variable.

3.1. Example 1

A homogeneous pinned–pinned Euler–Bernoulli beam with external constraints is considered as the first numerical example (lumped inertial, stiffness and damping elements) as proposed in Ref. [7]. Clearly, if a lumped damping element is considered, then the beam becomes non-proportionally damped. System parameters are chosen as follows:

- beam mass density $m_1 = m_2 = m = 1.6363 \times 10^4$ kg/m;
- beam bending stiffness $k_1 = k_2 = k = 1.6669 \times 10^{11}$ Nm^2;
- beam length $l = 15.24$ m;
- added mass $M_1 = 0.1\, ml$ at position $l_1 = l/2$;
- added translational spring with $K_1 = 0.1\, ml\omega_1^2$ at position $l_1 = l/2$ (where ω_1 is the first natural frequency of the beam without added mass and spring);
- added viscous damper with $C_1 = 0.1\, ml\omega_1$at position $l_1 = l/2$.

The continuity of displacement, rotation, moment and shear in $x = x_1$ is stated by Eq. (24), where matrix $\mathbf{B}_{p-1}$, Eq. (25), reduces to

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 0 & 0 & -k^{-1}r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{37}$$

being $r = M_1 s^2 + C_1 s + K_1$.

In Table 1, the first three eigenvalues obtained from Eq. (34) are compared with those computed in Ref. [7]: an excellent agreement is achieved. Note that the second mode is not affected by the presence of the added mass, spring and damper, due to their position, which is coincident with the node of the second undamped mode.

Table 1
Eigenvalues s_n rad/s of example Case 1

Mode (n)	Present study	[7]
1	$-1.130627e+001 \pm 1.351799e+002i$	$-1.130626e+001 \pm 1.351799e+002i$
2	$0 \pm 5.425144e+002i$	$-4.177324e-012 \pm 5.425147e+002i$
3	$-8.482803e+000 \pm 1.128716e+003i$	$-8.482803e+000 \pm 1.128716e+003i$

Table 2
Example Case 2 system parameters

Segment	Length (m)	Mass density (kg/m)	Bending stiffness (N m^2)	Damping density (Ns /m^2)
1	0.1905	0.243	4.725	5×10^{-4}
2	0.035	0.243	4.725	50
3	0.0395	0.243	4.725	5×10^{-4}
4	0.035	0.243	4.725	50

Table 3
Eigenvalues s_n of example Case 2

Mode (n)	Eigenvalue s_n [rad/s]
1	$-5.720324e+001 \pm 1.626707e+002i$
2	$-3.205648e+001 \pm 1.078383e+003i$
3	$-3.681733e+001 \pm 3.022033e+003i$
4	$-1.997599e+001 \pm 5.923198e+003i$
5	$-1.818495e+001 \pm 9.792150e+003i$
6	$-2.861540e+001 \pm 1.462773e+004i$
7	$-2.496616e+001 \pm 2.043052e+004i$
8	$-2.417755e+001 \pm 2.720052e+004i$

3.2. Example 2

Example 2 concerns a homogeneous beam with a non-proportional external damping distribution consisting of four different homogeneous segments. Since matrix dimensions remain constant (i.e., 4×4), the present approach reduces the increase of computational effort that affects those methods in which the dimensions grow proportionally to the number N of segments [8].

An Euler–Bernoulli beam clamped at $x = 0$ and free at $x = l$ is chosen, with beam total length $l = 0.30$ m. The system parameters for each segment are summarized in Table 2.

Due to the absence of lumped constraints and internal damping, matrices $\mathbf{B}_{p-1}$, Eq. (25) with $p = 1, 2, 3, 4$, simply reduces to identity. In Table 3 the first eight eigenvalues obtained by Eq. (34) are presented.

Note that the real part of the eigenvalues is different for each mode as a direct consequence of the non-proportionality. On the contrary, in case of external damping operator proportional to the mass operator, i.e., $c(x) = \beta m(x)$, where β is a constant, $\mathrm{Re}[s_n]$ is the same for each mode. For example, the same beam has been considered under the effect of different levels of proportional damping:

- damping constant equal to the minimum of the non-proportional distribution, i.e., $c = 5 \times 10^{-4}\,\mathrm{Ns/m^2}$, leading to $\mathrm{Re}[s_n] = -1.0288 \times 10^{-3}\,\mathrm{rad/s}$;
- damping constant equal to the maximum of the non-proportional distribution, i.e., $c = 50\,\mathrm{Ns/m^2}$, leading to $\mathrm{Re}[s_n] = -1.0288 \times 10^{2}\,\mathrm{rad/s}$;
- damping constant equal to the integral mean value of the non-proportional distribution, i.e., $c = 11.67\,\mathrm{Ns/m^2}$, leading to $\mathrm{Re}[s_n] = -24.012\,\mathrm{rad/s}$.

Comparing the last value ($\mathrm{Re}[s_n] = -24.012\,\mathrm{rad/s}$) with those of Table 3, it is evident that such a non-proportional damping distribution produces more relevant effects on lower order modes than on higher ones: in particular, the first three modes decay rates increase significantly. Hence, an optimum damping distribution may be chosen to control particular (low order) modes.

In Fig. 2 time-dependent second mode displacements (Eq. (35)) are depicted, corresponding to arbitrary initial conditions. It shows that, if motion is asynchronous, there is no stationary point and the points having zero displacement fluctuate during motion. It should be noted that,

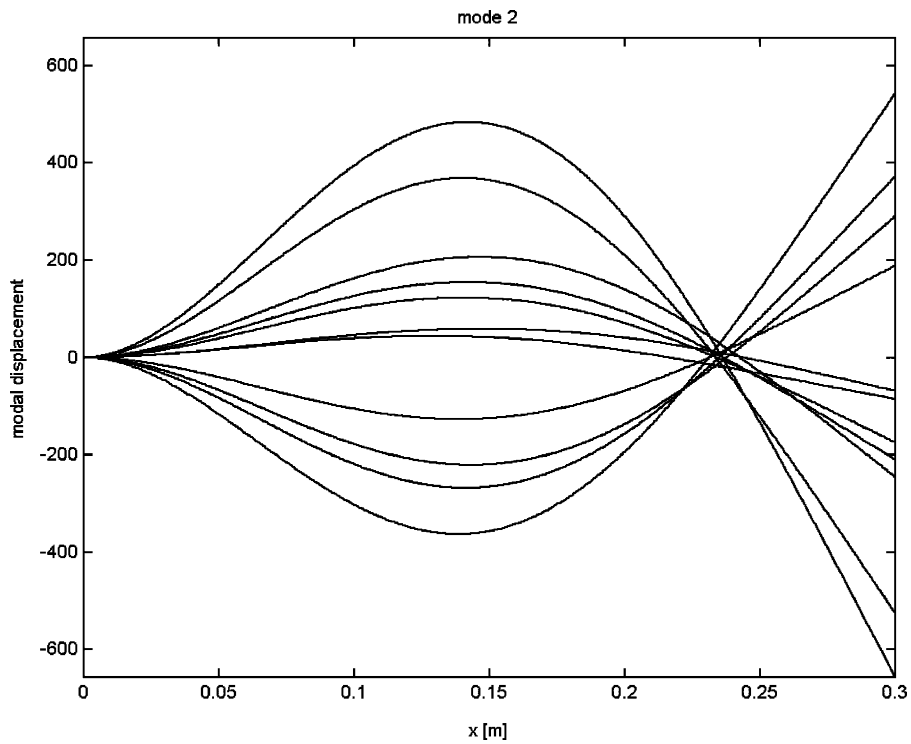

Fig. 2. Case 2: time-dependent modal displacement corresponding to the second pair of complex eigenfunctions.

although it is possible to define $\zeta_n = -\mathrm{Re}[s_n]/|s_n|$ even if damping is non-proportional, it does not correspond to a modal damping ratio in the classical sense, and it does not provide any information about the modal asynchronous behaviour: in fact, a relevant non-stationary modal displacement may correspond to a small ζ_n, as can be seen in Fig. 2, where $\zeta_2 = 3\%$ only.

3.3. Example 3

Example 3 concerns a homogeneous beam with non-proportional damping distributions consisting of two different homogeneous segments. An Euler–Bernoulli beam clamped at $x = 0$ and free at $x = l$ is considered, with total length $l = 0.30\,\mathrm{m}$ and external or internal damping distributions alternatively. The system parameters for each segment are as follows:

- length $l_1 = 0.20\,\mathrm{m}$, $l_2 = 0.10\,\mathrm{m}$;
- beam mass density $m_1 = m_2 = m = 0.243\,\mathrm{kg/m}$;
- beam bending stiffness $k_1 = k_2 = k = 4.725\,\mathrm{Nm}^2$.

In the following, different levels of non-proportionality are obtained by increasing the damping of a single segment, the other remaining unchanged. Differences between proportional and non-proportional damping effects are clearly outlined by means of root locus diagrams for the first three modes of the beam.

3.3.1. External distributed damping

In this example, the distributed external damping density relative to the first segment, $c_1 = 1.675\,\mathrm{Ns/m}^2$, is kept constant, while c_2 varies from $c_2 = c_1$ (proportional damping) to infinite (non-proportional damping limit case).

Fig. 3 shows a root locus comparison between proportional and non-proportional external damping for the first three modes of the beam. The curves relative to the proportional damping case are obtained by varying both c_1 and c_2 with $c_2 = c_1$. For each mode, the two trajectories (proportional and non-proportional case) start from the same point $s^{(prop)}$ corresponding to $c_2 = c_1 = 1.675\,\mathrm{N\,s/m}^2$. For the first mode no relevant difference can be observed between proportional and non-proportional external damping: the curves are nearly coincident, although they represent two different functions of c_1 and c_2. Otherwise, for the second and third mode the curves are nearly coincident in the neighbourhood of the starting point $s^{(prop)}$, then they strongly diverge at higher values of damping. In particular, the non-proportional behaviour for the second and third mode is substantially different: the second mode curve reaches the real axis for a certain value of c_2, i.e., the second mode becomes critically damped; on the other hand, the third mode curve never reaches the real axis, but it tends to $s_3^{(lim)}$ for $c_2 \to \infty$. The asymptotic behaviour of the third mode root locus can be explained by considering that as $c_2 \to \infty$, the clamped–free beam under analysis tends to transform into a clamped–clamped beam of total length l_1. As a consequence, the third mode of the initial configuration tends to the first mode (real) of the limit constraint set-up (see Fig. 4), while the first two modes vanish. In order to verify the accuracy of the proposed method, the value $s_3^{(lim)}$ has been successfully compared with the first eigenvalue of the clamped-clamped beam of total length l_1 (with proportional damping c_1) obtained by a classical technique [14] and also by the proposed method.

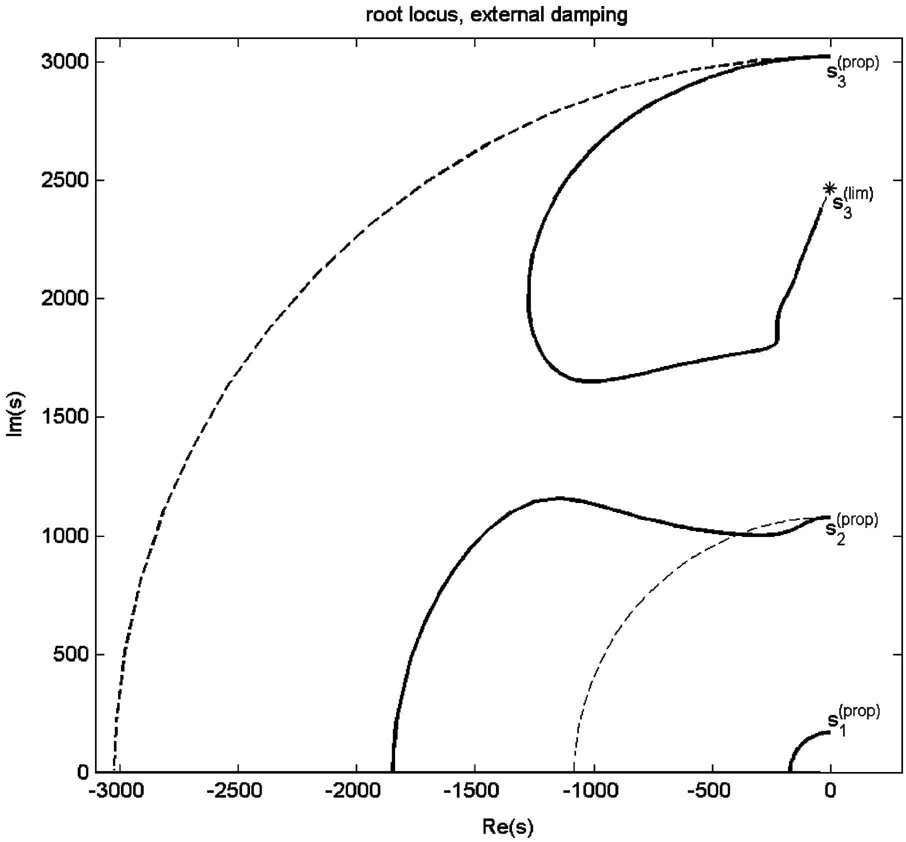

Fig. 3. Example 3: root locus for distributed external damping. Solid line: c_2 increases and c_1 remains unchanged (non-proportional damping); dashed line: both c_2 and c_1 increase with $c_2 = c_1$ (proportional damping).

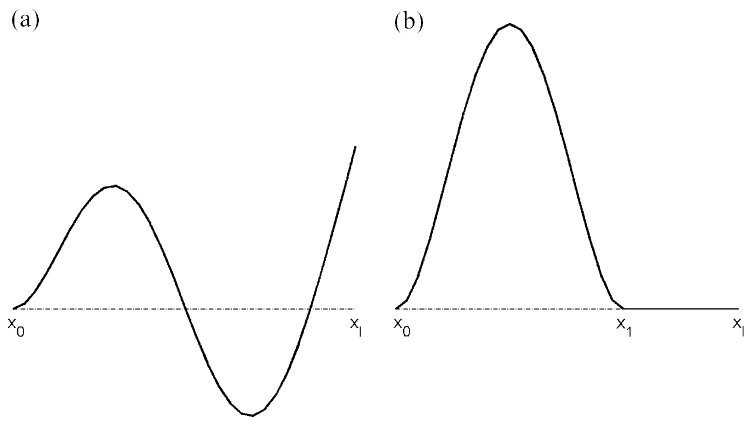

Fig. 4. Clamped–free beam with distributed external damping: modal displacement corresponding to the third pair of complex eigenfunctions; (a) $c_2 = c_1$ (proportional damping); (b) $c_2 \to \infty$.

3.3.2. Internal damping

In this example, differences between external and internal damping are in evidence, considering the same beam under the effect of an internal damping distribution, which is kept constant in the first segment, $c_{in,1} = 6.7 \times 10^{-5}\,\mathrm{N\,s\,m^2}$, while in the second one it varies from $c_{in,2} = c_{in,1}$ (proportional damping) to infinite (non-proportional damping limit case).

In Fig. 5 a root locus for the first three modes of the beam with non-proportional internal damping is depicted. If compared with the proportional case, the trajectory of each mode starts from the same point $s^{(prop)}$ corresponding to $c_{in,2} = c_{in,1} = 6.7 \times 10^{-5}\,\mathrm{N\,s\,m^2}$, it is nearly coincident only in a restricted region of the starting point, then it strongly diverges at higher values of damping and tends to $s^{(lim)}$ for $c_{in,2} \to \infty$. As $c_{in,2}$ increases, no mode becomes overdamped, but, on the contrary, the clamped–free beam under analysis tends to transform into a clamped–free beam with the second segment l_2 of infinite flexural rigidity, i.e., a clamped–free beam of total length l_1 connected at its end to a rigid body of length l_2 and mass m_2 (see Fig. 6).

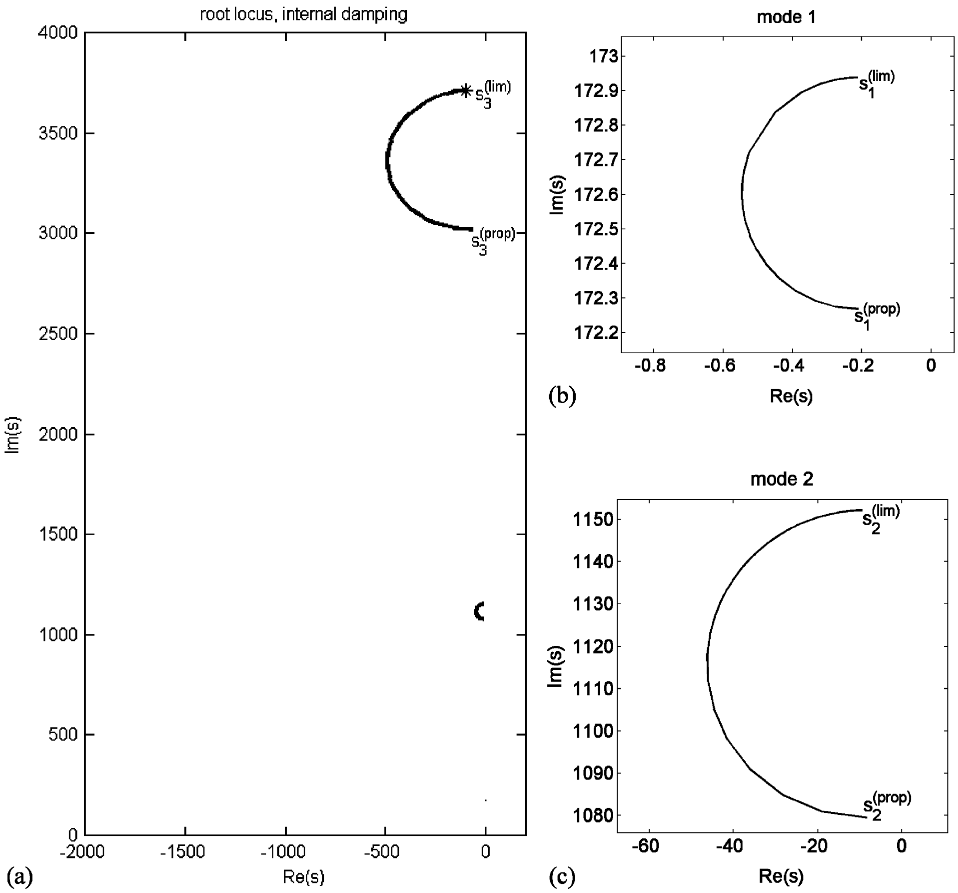

Fig. 5. Example 3: root locus for internal damping; $c_{in,2}$ increases and $c_{in,1}$ remains unchanged (non-proportional damping). (a) root locus for the first three modes; (b) zoom on first mode; (c) zoom on second mode.

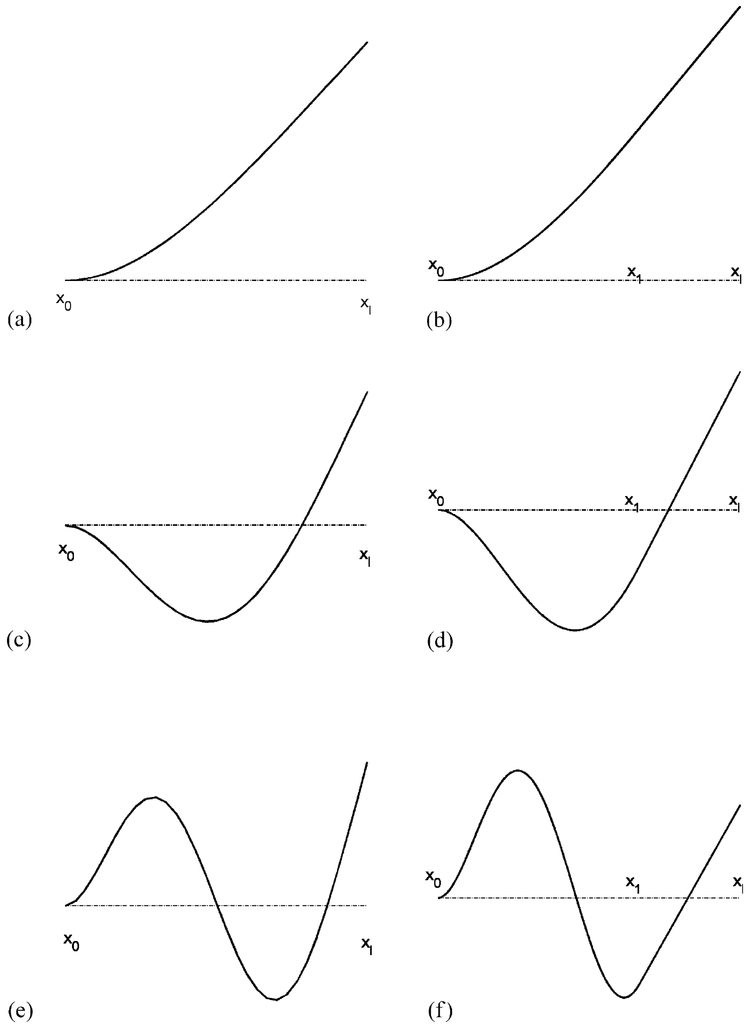

Fig. 6. Clamped–free beam with internal damping: modal displacement corresponding to a pair of complex eigenfunctions; left column $c_{in,2} = c_{in,1}$ (proportional damping); right column $c_{in,2} \to \infty$; (a) and (b) first mode; (c) and (d) second mode; (e) and (f) third mode.

Also in this example, the accuracy of the proposed method has been successfully tested. The values $s^{(lim)}$ have been compared with the first eigenvalues of the limit constraint set-up obtained by the proposed method restricted to the first segment. Note that in this case the boundary conditions matrix $\mathbf{B}_{el_1}$ becomes

$$\mathbf{B}_{el_1} = \begin{bmatrix} 1 & 0 & \dfrac{-m_2 l_2^2 s^2}{2(EI_1 + c_{in,1}s)} & \dfrac{-m_2 l_2 s^2}{EI_1 + c_{in,1}s} \\ 0 & 1 & \dfrac{m_2 l_2^3 s^2}{3(EI_1 + c_{in,1}s)} & \dfrac{m_2 l_2^2 s^2}{2(EI_1 + c_{in,1}s)} \end{bmatrix}, \tag{38}$$

due to shear and moment equilibrium equations of the rigid body of length l_2. This is an example of a boundary value problem with boundary conditions depending on the eigenvalues [12].

By comparing Figs. 3 and 5 some further considerations may apply:

- non-proportional damping distributions produce root locus curves very different from the circles due to proportional damping;
- the same non-proportional damping distribution leads to different root locus curves for different damping models (for example internal, external, etc.): that is not the case if damping is proportional;
- under the effect of non-proportional damping, some modes may not become overdamped and as damping tends to infinite they may transform into modes of a different system.

4. Conclusions and future work

An analytical method has been developed for the analysis of a class of vibrating continuous systems with non-proportional viscous damping distributions, according to different damping models. The proposed method has been implemented in particular for non-homogeneous Euler–Bernoulli beams in bending vibration; however, it can be easily applied also to strings, shafts, rods and Timoshenko beams with all possible boundary conditions. Complex modes, corresponding to non-synchronous motions, are obtained by a state-form expansion, applied together with a transfer matrix technique, resulting in easy computing implementation and high computational efficiency. Numerical examples show that non-proportional damping distributions induce relevant changes in root locus diagrams, depending also on the assumed damping model, and significant effects on the dynamic behaviour, corresponding to non-synchronous motions. Moreover, the numerical results confirm that optimum damping distributions may be chosen to control particular modes, and show that the damping ratio defined in the classical sense is scarcely informative when damping is non-proportional. Future effort may be devoted to making linear dynamic identification techniques able to take into account the effects due to non-proportionality, in particular in order to provide a correct estimate of natural frequencies and damping ratios. As a natural complement of the present study, the complex eigensolutions obtained will be the starting point of modal analysis, which will lead to the expression of impulse and frequency response functions and, consequently to enable comparisons with experimental data.

Appendix A. Nomenclature

a	transmission factor for a homogeneous beam segment
$\mathbf{B}$	boundary conditions matrix
b	coefficient of matrix **B** depending on flexural rigidity and internal damping
C	lumped damping
$\mathbf{c}$	vector of constants
c	external damping per unit length of beam
c_{in}	internal damping per unit length of beam
D	spatial domain

E	Young's modulus of elasticity
f	load per unit length of beam or force density
$\mathbf{I}$	identity matrix
I	area moment of inertia
i	imaginary unit
K	lumped stiffness
k	stiffness per unit length of beam or flexural rigidity
l	length of the beam
M	lumped mass
m	mass per unit length of beam
N	total number of steps of the spatial domain partition
q	normal co-ordinate
r	coefficient of matrix **B** depending on external constraints
$\mathbf{S}$	transposed companion matrix
s	complex eigenvalue
t	time
$\mathbf{w}$	state vector
w	transverse displacement
$\mathbf{x}, x$	spatial variables
$\mathbf{y}$	vector built with the eigenfunction ϕ and its first three derivatives with respect to x
Δx	beam segment length
$\boldsymbol{\Phi}$	eigenvector matrix of **S**
$\boldsymbol{\phi}$	eigenfunction
Γ	damping factor
γ	complex scaling factor
Λ	eigenvalue matrix of **S**
λ	eigenvalue of **S**
$\boldsymbol{\Pi}$	boundary conditions global transfer matrix
$\boldsymbol{\Theta}$	coefficient matrix of the algebraic boundary conditions system
σ	coefficient of the fourth-order differential equation for a homogeneous beam segment
Ω	angular frequency of damped free vibration
ω	angular frequency
ζ	damping ratio

Operators

$\mathrm{C}[\cdot]$	damping operator
$\mathrm{Im}[\cdot]$	imaginary part
$\mathrm{K}[\cdot]$	stiffness operator
$\mathrm{L}[\cdot]$	linear operator
$\mathrm{M}[\cdot]$	mass operator
$\mathrm{Re}[\cdot]$	real part

Subscripts

e	ends of the beam
n	modal index
p	spatial domain partition index

Superscripts

$*$	complex conjugate
I–IV	first to fourth derivative with respect to x
(w)	translational degree of freedom
(ϑ)	rotational degree of freedom

References

[1] U. Prells, M.I. Friswell, A measure of non-proportional damping, Mechanical Systems and Signal Processing 14 (2000) 125–137.

[2] S. Naguleswaran, Transverse vibrations of an Euler-Bernoulli uniform beam carrying two particles in-span, International Journal of Mechanical Sciences 43 (2001) 2737–2752.

[3] M.A. De Rosa, P.M. Belles, M.J. Maurizi, Free vibrations of stepped beams with intermediate elastic supports, Journal of Sound and Vibration 181 (1995) 905–910.

[4] S. Kukla, Free vibrations of axially loaded beams with concentrated masses and intermediate elastic supports, Journal of Sound and Vibration 172 (1994) 449–458.

[5] P.D. Cha, Alternative formulations to obtain the eigensolutions of a continuous structure to which spring-mass systems are attached, Journal of Sound and Vibration 246 (2001) 741–750.

[6] K.B. Subrahmanyam, A.K. Garg, Uncoupled flexural vibrations of straight beams with all possible boundary conditions treated by a transfer matrix method, Journal of Sound and Vibration 204 (1997) 397–419.

[7] T.-P. Chang, F.-I. Chang, M.-F. Liu, On the eigenvalues of a viscously damped simple beam carrying point masses and springs, Journal of Sound and Vibration 240 (2001) 769–778.

[8] G. Prater, R. Singh, Eigenproblem formulation, solution and interpretation for non-proportionally damped continuous beams, Journal of Sound and Vibration 143 (1990) 125–142.

[9] T.K. Hasselman, A. Kaplan, Dynamic analysis of large systems by complex mode synthesis, Journal of Dynamic Systems, Measurements, and Control, Transactions of the American Society of Mechanical Engineers 96 (1974) 327–333.

[10] R.R. Craig, Y.-T. Chung, Generalized substructure coupling procedure for damped systems, American Institute of Aeronautics and Astronautics Journal 20 (1982) 442–444.

[11] A.D. Nashif, D.I.G. Jones, J.P. Henderson, Vibration Damping, Wiley, New York, 1985.

[12] L. Meirovitch, Principles and Techniques of Vibrations, Prentice-Hall, Upper Saddle River, NJ, 1997.

[13] J.M. Ortega, W.C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970.

[14] R.D. Blevins, Formulas for Natural Frequency and Mode Shape, Van Nostrand Reinhold, New York, 1979.